

IRREDUCIBLE COMPONENTS OF THE NILPOTENT CONE OF THE TADPOLE QUIVER

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ABSTRACT. We exhibit the irreducible components of the nilpotent cone of the tadpole quiver of arbitrarily high codimension and introduce an approach that could allow to describe all of them.

INTRODUCTION

This note deals with the nilpotent cone of the tadpole quiver, which is one of the simplest quivers with loops. The goal is to find a description for the irreducible components of the nilpotent cone. According to G. Lusztig, the nilpotent cone of a quiver without loops is equidimensional (lagrangian). According to A. Khoroshkin (unpublished), the nilpotent cone of the tadpole quiver has some irreducible components of dimension less than expected. In this note we construct a series of examples of irreducible components of arbitrarily high codimension.

The definitions of the nilpotent cone and tadpole quiver are given in Section 1. Section 2 deals with a useful property of the equations defining the nilpotent cone, which allows to describe it as a certain fiber product of relatively simple manifolds. This description, along with a stratification arising from Jordan stratification on the space of nilpotent matrices, is given in the next section and is shown to be useful in the study of irreducible components. Section 4 describes an upper bound on the number of components, effectively reducing the problem to the description of a smaller variety. While the complete description is unknown, this result allows one to construct some of the irreducible components of this nilpotent cone. Two different cases corresponding to the highest and the lowest elements of Jordan stratification are considered in Section 5, giving the components of maximal and minimal possible codimension respectively. In particular, an example of the components of arbitrarily high codimension is described.

1. DEFINITIONS

We recall some material of [2]. Let Γ be a finite graph, with I being the set of vertices and E being the set of edges. Let H be the set consisting of edges along with a choice of orientation on them, then an orientation of Γ is a subset $\Omega \subset H$, such that $\overline{\Omega} \cap \Omega = \emptyset$ and $\overline{\Omega} \cup \Omega = H$, where the bar denotes the inverted orientation. The choice of orientation is equivalent to the choice of function $\omega : H \rightarrow$

$\{\pm 1\}$, such that $\omega(\bar{h}) = -\omega(h)$, the correspondence given as follows: $\Omega = \{h \in H | \omega(h) = 1\}$. For $h \in H$, let $in(h)$ and $out(h)$ denote its incoming and outgoing vertices respectively.

The quiver is a finite graph Γ with an orientation Ω . Most of the constructions below do not, in fact, depend on the choice of orientation, so we will often refer to a graph as "quiver" without specifying the orientation, thus meaning that any orientation will do. Particular choice of orientation will be required mostly for notational purposes.

Definition 1. *The tadpole quiver T_n is a quiver with n vertices $1, 2, \dots, n$ and n edges h_1, h_2, \dots, h_n , with orientation chosen, such that $in(h_i) = i$, $out(h_i) = i + 1 \ \forall 1 \leq i \leq n - 1$ and $in(h_n) = out(h_n) = n$.*

In other words, it's a quiver of type A_n with an additional loop at the last vertex.

For arbitrary quiver (Γ, Ω) , fix a dimension vector $v \in \mathbb{Z}_{\geq 0}^I$ and a set of vector spaces $\{V_i | i \in I, \dim(V_i) = v_i\}$. Consider the space $M_{\Gamma, v} = \bigoplus_{h \in H} \text{Hom}(V_{in(h)}, V_{out(h)})$. For an arbitrary element $B \in M_{\Gamma, v}$, $B = \bigoplus_{h \in H} B_h$.

Definition 2. *An element $B \in M_{\Gamma, v}$ is called nilpotent, if $\exists N \in \mathbb{N}$, such that for any path $h_1 h_2 \dots h_k$, $out(h_i) = in(h_{i+1})$, $k > N$, the composition $B_{h_k} B_{h_{k-1}} \dots B_1 = 0$.*

Definition 3. *The moment map $\mu : M_{\Gamma, v} \rightarrow \bigoplus_{i \in I} \mathfrak{gl}(V_i)$ is given by the following formula:*

$$\mu(B)_i = \sum_{h \in H: in(h)=i} \omega(h) B_{\bar{h}} B_h$$

It is obvious from this formula that μ depends on the orientation. Change of orientation is equivalent, as far as the moment map is concerned, to multiplication by -1 in several matrix summands of $M_{\Gamma, v}$, which does not change the resulting varieties.

Definition 4. *The subvariety*

$$\mathfrak{N}_{\Gamma, v} = \{B \in M_{\Gamma, v} | \mu(B) = 0, B \text{ is nilpotent}\} \subset M_{\Gamma, v}$$

is called nilpotent cone corresponding to the quiver Γ .

2. NILPOTENCY CONDITION

It is well known that the definition of the nilpotent cone is, in fact, redundant. To see this let us first consider a quiver of type A_n with the edges being oriented in one direction. Denoting the vertices $1, 2, \dots, n$, and the edges h_1, h_2, \dots, h_{n-1} , such that $in(h_i) = i$, we can write the condition $\mu = 0$ as a set of equations $B_{\bar{h}_1} B_{h_1} = 0$, $B_{h_{n-1}} B_{\bar{h}_{n-1}} = 0$, $B_{h_{i-1}} B_{\bar{h}_{i-1}} = B_{\bar{h}_i} B_{h_i} \ \forall 2 \leq i \leq n - 1$. This means that, first, for any path containing a fragment $\dots h_i \bar{h}_i \dots$ the resulting product as in

the definition of the nilpotent element coincides with the product for the path obtained by replacing this fragment by $\dots \overline{h_{i-1}} h_{i-1} \dots$, and second, this product automatically equals to 0 for any path passing through the extreme vertices. It follows that the condition $\mu = 0$ implies the nilpotency condition in this case, with $N = 2(n - 2) \forall B \in \mu^{-1}(0)$.

Now let us consider a finite connected graph Γ with cycles generated by loops only, such that all vertices save one are of degree less than or equal to 2. This means, equivalently, that this graph consists of several type A quivers and several loops linked to one initial vertex, a special case of this being the tadpole quiver. Fix an orientation Ω . Denote the initial vertex i_0 , and denote $A = \{h \in H | in(h) = i_0, out(h) \neq i_0\}$, $L = \{h \in \Omega | in(h) = out(h) = i_0\}$. As before, let V and v be the set of vector spaces corresponding to the vertices and their dimension vector respectively. Consider the following operators N_h , $h \in A \cup L$, in $End(V_{i_0})$: $\forall h \in A$, $N_h = \omega(h) B_{\overline{h}} B_h$; $\forall h \in L$, $N_h = [B_{\overline{h}}, B_h]$. The condition $\mu = 0$ is the same as in the case of type A quiver for all vertices save the initial one, and for the latter it can be written as $\sum_{h \in A \cup L} N_h = 0$.

Definition 5. *The set of operators $X_j \in End(V)$, $j \in J$ for any vector space V and any index set J is called nilpotent, if $\exists N \in \mathbb{N}$ such that any composition of k operators $X_j \forall k > N$ is equal to 0.*

Proposition 1. *Under these assumptions, $B \in \mu^{-1}(0)$ is nilpotent iff $\{N_h | h \in A\} \cup \{B_h | h \in L \cup \overline{L}\}$ is nilpotent.*

Proof. To prove that $B \in \mu^{-1}(0)$ is nilpotent it suffices to prove that the composition is 0 for any sufficiently long path beginning and ending in the initial vertex. Any such path can be divided into elementary parts beginning and ending in the initial vertex and not passing through it. Due to the conditions $\mu = 0$ for the vertices of quiver of type A_n , the composition of operators from B along any such elementary path would equal to either B_h for $h \in L \cup \overline{L}$ or N_h^k for some $h \in A$, $k \in \mathbb{N}$. Thus the condition that $\{N_h | h \in A\} \cup \{B_h | h \in L \cup \overline{L}\}$ is nilpotent implies $B \in \mu^{-1}(0)$ is nilpotent. The second part of the statement is straightforward. \square

Proposition 1 states that the nilpotency condition is essentially local, that is, the only restrictions concern adjacent vertices. In effect, it allows to construct the nilpotent cone for such quivers as a fiber product of a set of relatively simple varieties corresponding to the edges of a quiver over another set of varieties corresponding to the vertices.

3. DECOMPOSITION

Let U be a vector space. Denote $\mathcal{N}(U)$ the variety of nilpotent endomorphisms of U . Let U_1, U_2 be two vector spaces, denote $\mathfrak{H}(U_1, U_2) = \{(h, \overline{h}) | h \in Hom(U_1, U_2), \overline{h} \in Hom(U_2, U_1), \overline{h}h \in$

$\mathcal{N}(U_1), h\bar{h} \in \mathcal{N}(U_2)\}$. Note that there exist natural morphisms $p_i: \mathfrak{H}(U_1, U_2) \rightarrow \mathcal{N}(U_i)$, $i = 1, 2$. Finally, for a vector space U , denote $\mathfrak{X}(U) = \{(h, \bar{h}) | h \in \text{End}(U), \bar{h} \in \text{End}(U), \text{the pair } (h, \bar{h}) \text{ is nilpotent}\}$. There is a morphism $\phi: \mathfrak{X}(U) \rightarrow \mathcal{N}(U)$, $\phi(h, \bar{h}) = [h, \bar{h}]$.

Proposition 2. *The variety $\mathfrak{N}_{T_n, v}$ is isomorphic to the fiber product $\mathfrak{H}(0, V_1) \times_{\mathcal{N}(V_1)} \mathfrak{H}(V_1, V_2) \times_{\mathcal{N}(V_2)} \dots \times_{\mathcal{N}(V_{n-1})} \mathfrak{H}(V_{n-1}, V_n) \times_{\mathcal{N}(V_n)} \mathfrak{X}(V_n)$.*

Proof. This is a direct corollary of Proposition 1 and the condition $\mu = 0$. For any $B \in \mathfrak{N}_{T_n, v}$ let $A_i = B_{\bar{h}_i} B_{h_i} \in \text{End}(V_i) \forall i < n$, $A_n = [B_{h_n}, B_{\bar{h}_n}]$. Then $A_i \in \mathcal{N}(V_i)$, $A_1 = 0$, $B_{h_i} B_{\bar{h}_i} = A_{i+1} \forall i < n$, and the triple $(B_{h_n}, B_{\bar{h}_n}, A_n)$ is nilpotent, hence $\mathfrak{N}_{T_n, v} \subset \mathfrak{H}(0, V_1) \times_{\mathcal{N}(V_1)} \mathfrak{H}(V_1, V_2) \times_{\mathcal{N}(V_2)} \dots \times_{\mathcal{N}(V_{n-1})} \mathfrak{H}(V_{n-1}, V_n) \times_{\mathcal{N}(V_n)} \mathfrak{X}(V_n)$. The opposite inclusion is again straightforward. \square

Remark. Note that similar decomposition exists for any quiver of the type considered in the previous section. One important example is A_n . The same arguments as before show that $\mathfrak{N}_{A_n, v}$ is isomorphic to the product $\mathfrak{H}(0, V_1) \times_{\mathcal{N}(V_1)} \mathfrak{H}(V_1, V_2) \times_{\mathcal{N}(V_2)} \dots \times_{\mathcal{N}(V_{n-1})} \mathfrak{H}(V_{n-1}, V_n) \times_{\mathcal{N}(V_n)} \mathfrak{H}(V_n, 0)$.

For a vector space U of dimension d consider the variety $\mathcal{N}(U)$. It admits stratification by subvarieties $\mathcal{N}_\lambda(U) \subset \mathcal{N}(U)$, $\forall \lambda \vdash d$, where $\mathcal{N}_\lambda(U)$ is a locally closed subset of all nilpotent operators A with Jordan form corresponding to the partition λ , that is, the sizes of Jordan blocks of A are equal to the elements of partition or, equivalently, $\dim(\text{Ker}(A^i)) - \dim(\text{Ker}(A^{i-1})) = (\lambda^T)_i$, where λ^T is a transposed partition.

The map $\phi: \mathfrak{X}(U) \rightarrow \mathcal{N}(U)$ described earlier allows to introduce similar stratification $\mathfrak{X}_\lambda(U) = \phi^{-1}(\mathcal{N}_\lambda(U)) \subset \mathfrak{X}(U)$. Note that ϕ is not surjective. In particular, $\mathfrak{X}_\lambda(U) = \emptyset$ iff λ has more than $\frac{d+1}{2}$ columns.

Now let U_1, U_2 be two vector spaces of finite dimensions d_1, d_2 respectively. The stratifications of the varieties $\mathcal{N}(U_i)$ along with two natural morphisms p_i lead to a stratification $\mathfrak{H}_{\lambda, \mu}(U_1, U_2) = p_1^{-1}(\mathcal{N}_\lambda(U_1)) \cap p_2^{-1}(\mathcal{N}_\mu(U_2)) \subset \mathfrak{H}(\mathcal{N}(U_1), \mathcal{N}(U_2)) \forall \lambda \vdash d_1, \mu \vdash d_2$.

The same is applicable to the quiver varieties. In case of quiver Γ of type A_n or T_n , the operators $B_{\bar{h}_i} B_{h_i} \in \text{End}(V_i)$ give a map $\mathfrak{N}_{\Gamma, v} \rightarrow \mathcal{N}(V_i)$ and the preimages of the strata $\mathcal{N}_{\lambda_i}(V_i)$ form a stratification $\mathfrak{N}_{\Gamma, \Lambda} \subset \mathfrak{N}_{\Gamma, v}$ itself, for $\Lambda = (\lambda^1 \vdash v_1, \lambda^2 \vdash v_2, \dots, \lambda^{n-1} \vdash v_{n-1}, \lambda^n \vdash v_n)$. For most Λ , however, the corresponding stratum turns out to be empty. For example, as $B_{\bar{h}_1} B_{h_1} = 0$, the stratum is empty if $\lambda^1 \neq (1^n)$, and for A_n the same holds for λ^n . Note that the description of the nilpotent cones as fiber products is compatible with Jordan stratification, in particular, the stratum $\mathfrak{N}_{\Gamma, \Lambda}$ is isomorphic to the fiber product of the strata of the factors over the corresponding strata of $\mathcal{N}(V_i)$, and it is empty if and only if at least one factor is empty. Let us call such a set of partitions Λ a *multipartition* of v , and write $\Lambda \vdash v$.

Definition 6. If $\lambda \vdash d_1$ and $\mu \vdash d_2$ are two partitions, a proper pairing between λ and μ is a correspondence between some of the elements of the partitions, such that any element $\lambda_i \geq 2$ of a partition, corresponds to an element μ_j of another partition, such that $|\lambda_i - \mu_j| \leq 1$, and elements equal to 1 may not have a pair.

The geometric significance of this notion is as follows: $\mathfrak{H}_{\lambda,\mu}(U_1, U_2) \neq \emptyset$ if and only if there exists a proper pairing between λ and μ . This follows easily from the classification of irreducible nilpotent representations of Jordan quiver. This implies that $\mathfrak{N}_{A_n,\Lambda} \neq \emptyset$ if and only if $\forall i$ there exists a proper pairing between λ^i and λ^{i+1} . Let us call a multipartition satisfying this condition *admissible*. In particular, for any admissible Λ , λ^1 and λ^n should contain only one column, as it is the only partition admitting a proper pairing with an empty partition of 0.

Let us denote $I_{\lambda,\mu}^{\mathfrak{H}}(U_1, U_2)$ the set of irreducible components of $\mathfrak{H}_{\lambda,\mu}(U_1, U_2)$, denote $I_{\lambda}^{\mathfrak{X}}(U)$ the set of irreducible components of $\mathfrak{X}_{\lambda}(U)$ and denote $I_{\Gamma,\Lambda}^{\mathfrak{N}}$ the set of irreducible components of $\mathfrak{N}_{\Gamma,\Lambda}$. Note that $I_{A_n,\Lambda}^{\mathfrak{N}}$ is in natural bijection with $\prod_{i=1}^{n-1} I_{\lambda^i,\lambda^{i+1}}^{\mathfrak{H}}(V_i, V_{i+1})$, and the dimension of $\gamma \in I_{A_n,\Lambda}^{\mathfrak{N}}$ is equal to $\sum_{i=1}^{n-1} \dim \gamma_i - \sum_{i=1}^n \dim \mathcal{N}_{\lambda^i}(V_i)$, where $\gamma_i \in I_{\lambda^i,\lambda^{i+1}}^{\mathfrak{H}}(V_i, V_{i+1})$ are the corresponding irreducible components of the factors. In the same way, $I_{T_n,\Lambda}^{\mathfrak{N}} = \left(\prod_{i=1}^{n-1} I_{\lambda^i,\lambda^{i+1}}^{\mathfrak{H}}(V_i, V_{i+1}) \right) \times I_{\lambda^n}^{\mathfrak{X}}(V_n)$, and the dimension of every component is equal to the sum of dimensions of the corresponding components of factors minus $\sum_{i=1}^n \dim \mathcal{N}_{\lambda^i}(V_i)$.

Note that every irreducible component of $\mathfrak{N}_{\Gamma,v}$ is a closure of an irreducible component of one of $\mathfrak{N}_{\Gamma,\Lambda}$, and every irreducible component of $\mathfrak{N}_{\Gamma,\Lambda}$ belongs to an irreducible component of $\mathfrak{N}_{\Gamma,v}$. It allows one to derive the number and dimensions of irreducible components of $\mathfrak{H}_{\lambda,\mu}(U_1, U_2)$ from A_n type quivers, and could lead to the description of the number and dimensions of irreducible components of $\mathfrak{N}_{T_n,v}$ if one can find them for $\mathfrak{X}_{\lambda}(U)$.

Proposition 3. $\forall \mu \vdash d_1, \eta \vdash d_2$ the irreducible components of $\mathfrak{H}_{\mu,\eta}(U_1, U_2)$ have the same dimension $d_1 d_2 + \frac{1}{2} \dim \mathcal{N}_{\mu}(U_1) + \frac{1}{2} \dim \mathcal{N}_{\eta}(U_2)$. The number of these components is denoted $\chi(\mu, \eta)$.

Proof. If λ is a partition, let $t(\lambda)$ be a partition obtained by omitting the first column of λ . Note that there exists a proper pairing between μ and η if and only if $t(\mu) \subset \eta$ and $t(\eta) \subset \mu$ as Young diagrams.

Recall that all irreducible components of $\mathfrak{N}_{A_n,v}$ have the same dimension equal to $\sum_{i=1}^{n-1} v_i v_{i+1}$.

If μ and η are two partitions with a proper pairing, consider a quiver A_n for $n = \mu_1 + \eta_1$, dimension vector v and a multipartition $\Lambda \vdash v$, such that $\lambda^i = t^{\mu_1-i}(\lambda)$ for $i \leq \mu_1$ and $\lambda^i = t^{i-\mu_1-1}(\eta)$ for $i > \mu_1$. For such data $\mathfrak{N}_{A_n,\Lambda}$ is a maximal nonempty element of Jordan

filtration on $\mathfrak{N}_{A_n, v}$, in particular, it is nonempty open subset of the latter, and the closures of its irreducible components are irreducible components of $\mathfrak{N}_{A_n, v}$, hence they are equidimensional. Thus, as $I_{A_n, \Lambda}^{\mathfrak{N}} = \prod_{i=1}^{n-1} I_{\lambda^i, \lambda^{i+1}}^{\mathfrak{N}}(V_i, V_{i+1})$, irreducible components of $\mathfrak{N}_{\mu, \eta}(V_{\mu_1}, V_{\mu_1+1})$ are also equidimensional. The dimension is easily obtained by induction, starting from $\dim \mathfrak{N}_{(1^{d_1}), (1^{d_2})}(V_1, V_2) = d_1 d_2$. \square

4. NUMBER OF COMPONENTS

Let us modify the notion of admissible multipartition to describe irreducible components of $\mathfrak{N}_{T_n, v}$. A multipartition is called (μ, η) -admissible, if $\lambda^1 = \mu, \lambda^n = \eta$ and $\forall i, 1 \leq i \leq n-1$, there exists a proper pairing between λ^i and λ^{i+1} . Let $\mathcal{A}_v(\mu, \eta)$ be a set of all (μ, η) -admissible $\Lambda \vdash v$. Let $\chi(\Lambda) = \prod_{i=1}^{n-1} \chi(\lambda^i, \lambda^{i+1})$, for $\chi(\lambda, \mu)$ defined in Proposition 3. Finally, let $\psi(\lambda) = |I_{\lambda}^{\mathfrak{X}}(U)|$ for a vector space U of dimension $|\lambda|$.

The stratification introduced in the previous section along with representation of the nilpotent cone as fiber product gives a way to compute the number of irreducible components of the nilpotent cone $\mathfrak{N}_{\Gamma, v}$ as well as their dimensions.

If $\Gamma = A_n$, there is $K(v) = \sum_{\Lambda \in \mathcal{A}_v((1^{v_1}), (1^{v_n}))} \chi(\Lambda)$ components, their dimensions are all equal to $\sum_{i=1}^{n-1} v_i v_{i+1}$. On the other hand, it is well known that $K(v)$ is a Kostant partition function. It gives one a way to compute the values of χ : for any λ and μ , consider a quiver as in the proof of Proposition 3. For pairs of partitions (λ, μ) and (η, ν) , let $(\lambda, \mu) \geq (\eta, \nu)$ if $\lambda \succeq \eta$ and $\mu \succeq \nu$, where $\lambda \succeq \eta$ denotes usual dominance ordering if $|\lambda| = |\eta|$, and denotes $|\lambda| > |\eta|$ otherwise. $K(v) = \sum_{\Lambda \in \mathcal{A}_v((1^{v_1}), (1^{v_n}))} \chi(\Lambda)$, and for all Λ encountered in this equation $\forall i, (\lambda^i, \lambda^{i+1}) \leq (\lambda, \mu)$. Thus, if we know $\chi(\eta, \nu)$ for every pair $(\eta, \nu) < (\lambda, \mu)$, we can derive $\chi(\lambda, \mu)$ from this equation. There are finitely many such pairs (η, ν) , so this allows to compute these numbers recursively. For example, $\chi((1^a), (1^b)) = K((a, b)) = \min(a, b) + 1$; considering the quiver A_3 one can find that $\chi((2^a 1^b), (2^c 1^d)) = 0$ if $a + b < c$ or $c + d < a$, that is, when there is no proper pairing, and equals to $\min(a, b) + \min(a + b - c, c + d - a) + 1$ otherwise.

If $\Gamma = T_n$, the number of components of a stratum $I_{\Gamma, \Lambda}^{\mathfrak{N}}$ is equal to $\chi(\Lambda)\psi(\lambda^n)$, and the dimension of any such component is equal to the dimension of underlying component of $\mathfrak{X}_{\lambda^n}(V_n)$ plus $(\sum_{i=1}^{n-1} v_i v_{i+1}) - \frac{1}{2} \dim \mathcal{N}_{\lambda^n}(V_n)$. Any irreducible component of $\mathfrak{N}_{T_n, v}$ coincides with a closure of an irreducible component of some stratum, so this effectively gives an upper bound on the set of irreducible components, because an irreducible component of a stratum may belong to the closure of irreducible component of a higher stratum. In low-dimension cases it does not in fact occur: the irreducible components of the higher strata can be shown to have dimension not greater than that of the

components of a lower stratum, that is, $\frac{1}{2} \dim \mathcal{N}_{\lambda^n}(V_n)$ increases faster than the dimensions of the components of $\mathfrak{X}_{\lambda^n}(V_n)$. It is not known if it holds true in general, and the set $I_{\lambda^n}^{\mathfrak{X}}(V_n)$ is also unknown.

5. EXAMPLES

Note that the variety of pairs of commuting nilpotent matrices $\mathfrak{X}_{(1^d)}(U)$ is irreducible of dimension $d^2 - 1$. There is a proof of this fact in [1].

Let v_n be equal to 1 or 2. In this case $\phi \equiv 0$, the variety $\mathfrak{X}(V_n) = \mathfrak{X}_{(1^{v_n})}(V_n)$ is irreducible of dimension $v_n^2 - 1$, and $\mathfrak{N}_{T_n, v} \simeq \mathfrak{N}_{A_n, v} \times \mathfrak{X}(V_n)$. In particular, all irreducible components have the same dimension. If $v_n \geq 3$, $\mathfrak{X}(V_n)$ is no longer equal to $\mathfrak{X}_{(1^{v_n})}(V_n)$, but there exists closed embedding $\mathfrak{N}_{A_n, v} \times \mathfrak{X}_{(1^{v_n})}(V_n) \hookrightarrow \mathfrak{N}_{T_n, v}$. The dimension of image of this embedding coincides with the dimension of the nilpotent cone, so one obtains a set of equidimensional irreducible components corresponding to irreducible components of $\mathfrak{N}_{A_n, v}$.

Let $\mathfrak{A}_n(v) = \mathfrak{H}(0, V_1) \times_{\mathcal{N}(V_1)} \mathfrak{H}(V_1, V_2) \times_{\mathcal{N}(V_2)} \dots \times_{\mathcal{N}(V_{n-1})} \mathfrak{H}(V_{n-1}, V_n)$, and $\mathfrak{A}_{n, \lambda}(v) \subset \mathfrak{A}_n(v)$ is a stratification obtained from Jordan stratification of $\mathcal{N}(V_n)$.

Consider the case of tadpole quiver with n vertices. Let $v_i \geq 2i$, $v_n = 2n$ or $v_i \geq 2i - 1$, $v_n = 2n - 1$, $\lambda = (\lfloor \frac{v_n+1}{2} \rfloor, \lfloor \frac{v_n}{2} \rfloor)$. Note that $\mathfrak{X}_{\lambda}(v_n)$ is an open dense subset of $\mathfrak{X}(v_n)$, in particular, $\dim \mathfrak{X}_{\lambda}(v_n) = \frac{3}{2}v_n(v_n - 1)$. Note also that $\dim \mathcal{N}_{\lambda}(v_n) = 4\lfloor \frac{v_n}{2} \rfloor \lfloor \frac{v_n-1}{2} \rfloor$. Hence $\mathfrak{A}_{n, \lambda}(v) \times_{\mathcal{N}_{\lambda}(v_n)} \mathfrak{X}_{\lambda}(v_n)$

is an open subset in $\mathfrak{N}_{T_n, v}$ of dimension $\sum_{i=1}^{n-1} v_i v_{i+1} + v_n^2 - n$. It is nonempty due to the conditions on v , and its irreducible components are irreducible components of $\mathfrak{N}_{T_n, v}$ of codimension $n - 1$. The number of these components is equal to $\sum_{\Lambda \in \mathcal{A}_v((1^{v_1}), \lambda)} \chi(\Lambda)$.

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